# Math 246C Lecture 20 Notes

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## 1 Failure of the Riemann Mapping Theorem and Solving the $\overline{\partial}$-Equation

### 1.1 Failure of the Riemann mapping theorem in several complex variables

**Theorem 1.1** (Poincaré)**.** *Let $D = \{z \in \mathbb{C} : |z| < 1\}$, and let $D^2 = D_z \times D_w \subseteq \mathbb{C}^2$ be the unit bidisc. There is no biholomorphic map $D^2 \to B_2 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 < 1\}$, the unit ball in $\mathbb{C}^2$.*

**Remark 1.1.** The Riemann mapping theorem does not hold for domains in $\mathbb{C}^n$ for $n > 1$.

**Remark 1.2.** Intuition: $\partial D^2$ contains non-constant analytic discs (holomorphic $f : D \to \partial D^2$), while $\partial B_2$ does not.

*Proof.* Assume that there exists a biholomorphic map $f : D^2 \to B_2$. Write $f(z, w) = (f^1(z, w), f^2(z, w))$. Let $w_0 \in \partial D_w$, and let $w_n \in D$ be such that $w_n \to w_0$. Then for any $z \in D$, $(z, w_n) \to (z, w_0) \in \partial D^2$. Then $|f(z, w_n)| \to 1$ (here, we only use that $f$ is **proper**: for any compact $K \subseteq B_2$, $f^{-1}(K)$ is compact).

On the other hand, we have $g_n(z) := f(z, w_n) \in \mathrm{Hol}(D, \mathbb{C}^2)$ with $|g_n(z)| \leq 1$. By normal families, passing to a subsequence, we get $g_n \to g \in \mathrm{Hol}(D, \mathbb{C}^2)$ locally uniformly. We have $|g(z)| = 1$ for all $z \in D$.

We claim that $g(z)$ is constant. Write $g(z) = (g^1(z), g^2(z))$, where

$$|g^1(z)|^2 + |g^2(z)|^2 = 1 \qquad z \in D.$$

Apply $\partial_z$:

$$(\partial_z g^1)\overline{g^1} + (\partial_z g^2)\overline{g^2} = 0.$$

Apply $\partial_{\overline{z}}$:

$$|\partial_z g^1|^2 + |\partial_z g^2|^2 = 0.$$

So $\partial_z g^i = 0$, and we get the claim.

Thus, $f(z, w_n)$ converges to a constant so that $f'_z(z, w_n) \to 0$. Let $z = z_0 \in D$ be fixed, and consider $h(w) = f'_z(z_0, w) = (h^1(w), h^2(w)) \in \mathrm{Hol}(D, \mathbb{C}^2)$. Write by Cauchy's integral formula:

$$h^j(w) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f^j(\zeta, w)}{(\zeta - z_0)^2} \, d\zeta, \qquad |z_0| < r < 1.$$

$h$ is bounded in $D$, so the radial limits $\lim_{r\to 1} h(rw_0)$ exist for almost every $w_0 \in \partial D$. We have that $h(w_n) \to 0$ if $w_n \to w_0 \in \partial D$. It follows that $\lim_{r\to 1} h(rw_0) = 0$ for almost every $w_0 \in \partial D$, and by the uniqueness theorem, $h(w) \equiv 0$ for $|w| < 1$. We get that $f'_z(z, w) = 0$ for all $(z, w) \in D^2$, so $f = f(w)$. Replacing the role of $z$ and $w$, we get that $f$ is constant. $\square$

## 1.2 Solving the $\overline{\partial}$-equation with compactly supported right hand side

Recall that if $\varphi \in C_0^k(\mathbb{C})$ with $k \geq 1$ and we set

$$u(z) = -\frac{1}{\pi} \iint \frac{\varphi(\zeta)}{\zeta - z} \, L(d\zeta),$$

then $u \in C^k(\mathbb{C})$, and $\frac{\partial u}{\partial \overline{z}} = \varphi$.

**Remark 1.3.** In general, the equation $\frac{\partial u}{\partial \overline{z}} = \varphi$ has no solutions with compact support.

In $\mathbb{C}^n$, when $n > 1$, the $\overline{\partial}$-equation is a system:

$$\frac{\partial u}{\partial \overline{z}_j} = f_j, \qquad 1 \leq j \leq n.$$

This is an overdetermined system, which cannot be solved unless the right hand side satisfies the compatibility conditions

$$\frac{\partial f_j}{\partial \overline{z}_k} = \frac{\partial f_k}{\partial \overline{z}_j} \qquad 1 \leq j, k \leq n.$$

**Remark 1.4.** If we view $\overline{\partial} u = \sum_{j=1}^n \frac{\partial u}{\partial \overline{z}_j} \, d\overline{z}_j$ as a 1-form and introduce the 1-form $f = \sum_{j=1}^n f_j \, d\overline{z}_j$, then the system becomes

$$\overline{\partial} u = f.$$

If we define the 2-form $\overline{\partial} f = \sum_{j=1}^n \overline{\partial} f_j \wedge d\overline{z}$, then the compatibility conditions become $\overline{\partial} f = 0$:

$$\overline{\partial} f = \sum_{j=1}^n \left( \sum_{k=1}^n \frac{\partial f_j}{\partial \overline{z}_k} \, d\overline{z}_k \right) \wedge dz_j = \sum_{j<k} \left( \frac{\partial f_j}{\partial \overline{z}_k} - \frac{\partial f_k}{\partial z_j} \right) \, d\overline{z}_k \wedge d\overline{z}_j$$

**Theorem 1.2.** *Let $f_j \in C_0^k(\mathbb{C}^n)$ for $1 \leq j \leq n$ and $n > 1$ be such that $\overline{\partial} f = 0$. Then the equation $\overline{\partial} u = f$ has a solution $u \in C_0^k(\mathbb{C}^n)$.*

**Remark 1.5.** Such a solution is unique: if $u, \tilde{u} \in C_0^k(\mathbb{C}^n)$, then $\overline{\partial}(u - \tilde{u}) = 0$. So $u - \tilde{u} \in \mathrm{Hol}(\mathbb{C}^n)$ with compact support. So $u = \tilde{u}$.

*Proof.* Consider $\frac{\partial u}{\partial \bar{z}_j}$ for $1 \leq j \leq n$. Define

$$u(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{f_1(\zeta_1, z_2, \ldots, z_n)}{\zeta_1 - z_1} L(d\zeta_1) = -\frac{1}{\pi} \iint \frac{f_1(\zeta_1 + z_1, z_2, \ldots, z_n)}{\zeta_1} L(d\zeta_1).$$

Then $u \in C^k(\mathbb{C}^n)$, and $\frac{\partial u}{\partial \bar{z}_1} = f_1$. $\square$

We will continue the proof next time.